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THE DYNAMIC INDENTATION OF AN ELASTIC HALF-SPACE BY A RIGID PUNCH

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Abstract—The two-dimensional contact problem between a rigid die and an elastic half-space is considered. A numerical method of solution is proposed which involves an iterative process which is continued until the correct solution is obtained according to certain criteria. The method is general enough and can handle punches of arbitrary shape as well as time-dependent indentation velocities. The treatment is unified for subsonic, transonic and supersonic indentations. The numerical procedure is checked with analytical results which are known in several special cases and good agreement is obtained. Results are presented for the smooth as well as frictional indentation by a wedge-shaped die and for a smooth parabolic punch.

INTRODUCTION

In elastodynamic problems which involve the indentation of the surface of a solid by a rigid indentor, mixed boundary conditions over a time-dependent region are encountered. As the indentor is depressed into the solid, there are some points on the surface of the solid which at first have traction free boundary conditions but later have displacements boundary conditions in order that the boundary of the body confirm to the geometry of the indentor in the contact region. The time-dependent contact region is not known in advance (except in supersonic indentations) but must be determined from the process of the solution of the contact problem. Obviously, this fact severely complicates the problem and, as a result of this, analytical solutions are known only in some simple cases. A review on various contact problems can be found in [1].

For a wedge-shaped indentor pressed to an elastic half-space at a constant velocity, Robinson and Thompson[2] obtained the stress at the surface of the half-space as well as the contact velocity which is assumed to be constant. An extension to the indentation problem by a conical die is presented in [3]. Bedding and Willis[4, 5] solved the same problems for the case of perfect adhesion but not including the case of transonic indentation which is still an open problem. In [6] the subsonic frictionless indentations by a wedge and parabolic punch are treated. In all cases explicit expressions for the stress and displacement components at any point of the half-space are evidently very hard to obtain.

Generalization to the frictionless identation by a rigid punch of arbitrary shape was discussed in [7] but under the facilitating assumption that the contact expands in a supersonic speed. As a result of this assumption, the contact region is completely defined by the portion of the die which has crossed the original position of the surface of the half-space, so that it is known in advance.

Brock[8] studied the smooth indentation by a rigid indentor of arbitrary shape and varying indentation velocity. He assumed that the indentor shape and displacement history can be represented by polynomial curves but under the restriction that the contact region expands at a constant sub-Rayleigh speed.

In this paper we propose a numerical method of solution which is able to solve the two-dimensional dynamic contact problem of the half-space. The method is general enough and can treat problems involving punches of arbitrary shape as well as varying indentation velocity. The numerical algorithm is unified in the sense that it need not distinguish between subsonic, transonic or supersonic contact velocities. This is in contradistinction to the analytical treatments in which the mathematical attack of each case is different. Due to the generality of the procedure, problems in which time-dependent contact velocities are encountered can be also treated. This is in contrast to the previous analytical treatments where the contact velocity was assumed to be constant. In those problems which involve time-dependent velocities, the contact velocity might vary from the subsonic to the transonic and supersonic region. The cases of smooth as well as frictional indentations can be treated by the present numerical method with the same ease.

The numerical process is based on an iterative procedure which is continued until the correct solution is obtained. This solution is determined by the equations of motion, the moving mixed boundary conditions and the requirements that the contact stress beneath the punch is compressive and that no interpenetration can occur outside the contact region. The reliability of the method is checked in several situations where analytical results are known and good agreements are obtained.

Results are presented for smooth as well as frictional wedge-shaped punch and for a parabolic punch. The applicability of the method of solution to other problems is mentioned.

FORMULATION OF THE PROBLEM

Consider a homogeneous isotropic elastic half-space $y \geq 0$. The two-dimensional elastodynamic equations of motion, in the absence of body forces, are given by

$$\rho \frac{\partial^2}{\partial t^2} \mathbf{u} = (\lambda + 2\mu)\, \text{grad div } \mathbf{u} - \mu \text{ rot rot } \mathbf{u} \tag{1}$$

where $\mathbf{u}(x, y, t) = \{u, v\}$ is the displacement vector whose components $u(x, y, t)$ and $v(x, y, t)$ are in the x and y directions respectively and t is the time. In (1) λ, μ are the Lamé constants of the material and ρ its density. The compressional and shear wave speeds in the material are given respectively by $c_1 = [(\lambda + 2\mu)/\rho]^{1/2}$, $c_2 = [\mu/\rho]^{1/2}$.

The half-space is assumed to be initially at rest. At time $t = 0$ a rigid punch starts to indent the half-space at a velocity $V(t)$. It is assumed that the indentor is symmetrical about the y-axis (although the proposed method of solution can be extended to the non-symmetrical case). During the indentation the generally unknown contact region between the punch and the half-space varies with time.

The boundary conditions at the surface of the half-space in the case of a smooth indentation are

$$v = f(x, t), \quad \sigma_{yx} = 0 \quad \text{for} \quad |x| \leq X(t), \quad y = 0, \quad t > 0 \tag{2}$$

$$\sigma_{yx} = \sigma_{yy} = 0 \quad \text{for} \quad |x| > X(t), \quad y = 0, \quad t > 0 \tag{3}$$

In (2–3) σ_{ij} are the components of the stress tensor, $f(x, t)$ is the prescribed vertical displacement imposed over the time-dependent region of contact of the surface and $X(t)$ describes the position of the edge of the unknown moving region with $X(0) = 0$.

The appropriate boundary conditions when perfect adhesion between the indentor and the half-space is assumed (i.e. a slip at the interface is completely prevented) are given by (3) with

$$\frac{\partial u}{\partial t} = 0, \quad v = f(x, t) \quad \text{for} \quad |x| \leq X(t), \quad y = 0, \quad t > 0 \tag{4}$$

For a wedge-shaped die, see Fig. 1(a), the function $f(x, t)$ in (2) has the form

$$f(x, t) = p(t) - x \tan\theta \tag{5}$$

where $p(t)$ is the penetration distance along the y-axis of the die, i.e. $p(t) = v(x = 0, y = 0, t)$. If in addition the punch indents the half-space at a constant speed V, then $p(t) = Vt$ which is the problem considered in [2], [4], [5] and [6].

For a parabolic punch, see Fig. 1(b), the function $f(x, t)$ is of the form

$$f(x, t) = p(t) - bx^2 \tag{6}$$

where b is a parameter. The case of a uniformly accelerated frictionless indentation for which $p(t) = at^2/2$, such that the velocity of indentation is given by at, is considered in [6].

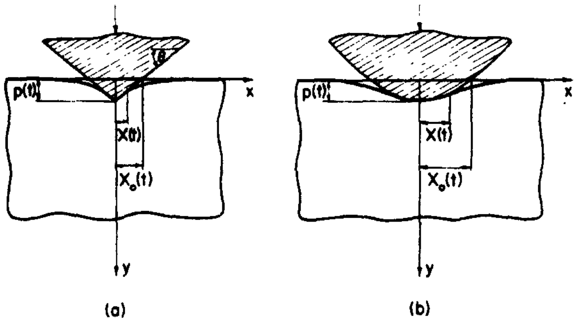

Fig. 1(a) A wedge-shaped punch. (b) A parabolic punch.

The principle difficulty in contact problems is the determination of the region of the contact $X(t)$ from which the contact velocity $\alpha(t) = \dot{X}(t)$ can be obtained. In all the analytical treatments, problems with constant α were considered. For constant indentation speed $V = \dot{p}$ of a wedge-shaped punch and for uniformly accelerated pressing of a parabolic punch the contact velocity α is constant.

In the following we present a numerical algorithm which enables us to solve the dynamic contact problem with various forms of $f(x, t)$ so that varying indentation and contact velocities as well as various shapes of punches can be treated. The problem of a wedge-shaped punch pressed with a constant speed as well as a parabolic punch pressed with a uniform acceleration are obtained as special cases.

NUMERICAL TREATMENT

The numerical solution of the dynamic contact problem which was formulated in the previous section can be divided into two parts.

(1) In the first part, a finite-difference approximation of the equations of motion (1) is performed by introducing a grid of mesh sizes Δx and Δy in the x and y directions respectively, together with a time increment Δt and replacing the derivatives of $\mathbf{u}(x, y, t)$ in (1) by their corresponding central difference expressions. Consequently, we obtain an explicit three-level difference scheme from which it is possible to compute the displacement vector at time $t + \Delta t$ whenever its values at the previous time steps at t and $t - \Delta t$ are known throughout the medium $y > 0$. Furthermore, the difference approximation is of a second order accuracy. The explicit form of the difference equations can be found in Ref. [9], and we need not describe it here. It was shown in [9] that the difference equations are stable with the stability condition

$$(c_1{}^2 + c_2{}^2)(\Delta t/\Delta x)^2 \leq 1 \tag{7}$$

for $\Delta x = \Delta y$.

(2) The second part of the numerical procedure consists of the treatment of the boundary conditions (2)–(4) taking into account the fact that the region of contact $0 \leq x \leq \dot{X}(t)$ is not known in advance, but should be determined from the requirements of the dynamic contact problem. These requirements are: (a) The normal stress σ_{yy} beneath the contact region of the punch with the half-space must be compressive. (b) No interpenetration outside the contact region can occur. This implies that the deformed position of the boundary outside the contact region must lie below the surface of the punch.

The numerical process for the treatment of the boundary conditions is based on the fact that if both these requirements are satisfied as well as the boundary conditions (2), (3) or (2) and (5), then the correct solution has been obtained. If these requirements are not satisfied then an iterative procedure, which is described in the following, should be continued.

We start the iterations with $X(t) = X_0(t)$ where $X_0(t)$ is the abscissa of the point at which the punch intersects the x axis at time t. For the wedge-shaped punch, for example, $X_0(t) = p(t)/\tan\theta$ (see Fig. 1(a)) and for the parabolic punch, Fig. 1(b), $X_0(t) = [p(t)/b]^{1/2}$. Having determined the edge of the region of contact we can impose the appropriate mixed boundary conditions.

The boundary conditions (2), (3) at the surface of the half-space $y = 0$ are imposed by introducing the explicit expressions for the stress components σ_{yx} and σ_{yy} in terms of the displacement gradients and approximating the latter by central difference expressions for derivatives in the x direction and by forward difference expressions for those in the y direction. Consequently, at every time step $t = n\Delta t$ $(n = 1, 2, \ldots)$ a system of algebraic equations in the unknown displacements at the boundary $y = 0$ is obtained.

Let us denote

$$\begin{aligned} \mathbf{u}_{i,j,n} &= \mathbf{u}(i\Delta x, j\Delta y, n\Delta t) = \mathbf{u}(x, y, t) \\ i &= 0, \pm 1, \pm 2, \ldots \\ j &= 0, 1, 2, \ldots \\ n &= 1, 2, \ldots \end{aligned} \tag{8}$$

Then the above system can be written in the form

$$\left.\begin{aligned} u_{i,0,n} - \epsilon[v_{i+1,0,n} - v_{i-1,0,n}] &= u_{i,1,n} \\ v_{i,0,n} &= f(i\Delta x, n\Delta t) \end{aligned}\right\} i = 0, 1, 2, \ldots, i_0 \tag{9}$$

$$\left.\begin{aligned} u_{i,0,n} - \epsilon[v_{i+1,0,n} - v_{i-1,0,n}] &= u_{i,1,n} \\ v_{i,0,n} - \epsilon\delta[u_{i+1,0,n} - u_{i-1,0,n}] &= v_{i,1,n} \end{aligned}\right\} i = i_0 + 1, i_0 + 2, \ldots \tag{10}$$

In the above equations for the unknown surface displacements $u_{i,0,n}$, $v_{i,0,n}$

$$i_0 = X_0(t)/\Delta x, \quad \epsilon = \Delta y/2\Delta x, \quad \delta = \lambda/(\lambda + 2\mu)$$

and we employ the fact that the punch is symmetric with the y-axis so that points $x < 0$ need not be considered.

It is obvious from (9) that we can immediately determine the surface displacements at the points $i = 0, 1, \ldots, i_0 - 1$. For $i = i_0, i_0 + 1, \ldots$ the system of equations are coupled. We found that it is very convenient and efficient, both from rate of convergence and programing points of view, to solve this system of equations by the Gauss–Seidel iterative procedure [10]. To this end, we notice that it is possible to split the equations and represent them in the form

$$\mathbf{Y}_r = \mathbf{E}_r\mathbf{Y}_r + F_r \quad r = 1, 2 \tag{11}$$

such that the first system ($r = 1$) is uncoupled to the second one ($r = 2$). In (11) $\mathbf{Y}_r$ are vectors of unknowns

$$\begin{aligned} \mathbf{Y}_1 &= \{u_{i_0,0,n}, \quad v_{i_0+1,0,n}, \quad u_{i_0+2,0,n}, \ldots\} \\ \mathbf{Y}_2 &= \{u_{i_0+1,0,n}, v_{i_0+2,0,n}, u_{i_0+3,0,n}, \ldots\} \end{aligned} \tag{12}$$

and $\mathbf{E}_r$ are coefficient matrices whose diagonal elements are all zero. Their rows contain the following typical non-zero elements

$$\begin{gathered} -\epsilon \\ -\epsilon, \epsilon \\ -\epsilon\delta, \epsilon\delta. \end{gathered} \tag{13}$$

Finally, the vectors $\mathbf{F}_r$ in (11) are

$$\begin{aligned} \mathbf{F}_1 &= \{u_{i_0,1,n} - \epsilon v_{i_0-1,0,n}, \quad v_{i_0+1,1,n}, \quad u_{i_0+2,1,n}, \ldots\} \\ \mathbf{F}_2 &= \{u_{i_0+1,1,n} - \epsilon v_{i_0,0,n}, \quad v_{i_0+2,1,n}, \quad u_{i_0+3,1,n}, \ldots\} \end{aligned} \tag{14}$$

According to the Gauss–Seidel iterative procedure we compute the mth approximation of the

lth component of $\mathbf{Y}_r$ by employing the mth approximations of the already computed q componts ($q < l$) of $\mathbf{Y}_r$. Thus

$$\mathbf{Y}_r^{(m)} = \mathbf{L}_r Y_r^{(m)} + \mathbf{U}_r \mathbf{Y}_r^{(m-1)} + \mathbf{F}_r \quad m = 1, 2, \ldots \tag{15}$$

where $\mathbf{L}_r$ and $\mathbf{U}_r$ are the lower and upper triangular matrices of $\mathbf{E}_r$, respectively. For $\mathbf{Y}_r^{(0)}$ we employ the values of $\mathbf{Y}_r$ in the previous time step $n-1$.

In order to show that the iterative process (15) converges, we employ theorem (3.4) in Varga[10], which asserts that if the matrices $\mathbf{I} - \mathbf{E}_r$ are irreducibly diagonally dominant then the iterative procedure (15) for the equations $(\mathbf{I} - \mathbf{E}_r)\,\mathbf{Y}_r = \mathbf{F}_r$ are convergent for any initial vectors $\mathbf{Y}_r^{(0)}$.

The matrices $\mathbf{I} - \mathbf{E}_r$ are indeed irreducible since it can be verified by examining the location of the elements (13) in $\mathbf{E}_r$ that their directed graphs[10] are strongly connected. This property expresses the fact that in each one of the system of matrix eqns (11), the equations are coupled and it is not possible to reduce any system to the solution of a lower order matrix equation.

It remains to show that the matrices $\mathbf{I} - \mathbf{E}_r$ are diagonally dominant, i.e.

$$\sum_m |(E_r)_{l,m}| \le 1 \quad l = 1, 2, \ldots \tag{16}$$

with strict inequality for at least one l. Referring to the typical rows of $\mathbf{E}_r$ given by (13), we obtain the following inequalities

$$\epsilon = \Delta y/2\Delta x \le 1 \quad \text{for} \quad \Delta y \le 2\Delta x,$$

$$2\epsilon = \Delta y/\Delta x \le 1 \quad \text{for} \quad \Delta y \le \Delta x,$$

$$2\epsilon|\delta| = (\Delta y/\Delta x)|\lambda/(\lambda + 2\mu)| = (\Delta y/\Delta x)\lambda/(\lambda + 2\mu) < 1. \tag{17}$$

In the last inequality, we have utilized the inequalities $\lambda + 2\mu/3 > 0$, $\mu > 0$ for the positive definiteness of the strain energy of an isotropic material, and also the relation $\lambda > 0$ for real materials. Consequently, the matrices $\mathbf{I} - \mathbf{E}_r$ are diagonally dominant for $\Delta y \le \Delta x$ with strict inequality in (16) for at least one l. Hence the iterative procedure (15) is convergent.

Having computed all the displacements at the boundary $y = 0$ of the half-space, we can calculate the stresses within the assumed contact region and verify that the previous two requirements for the dynamic contact problem are satisfied. If the answer is affirmative, we deduce that the correct solution at time $t = n\Delta t$ has been obtained so that we can proceed to the next time step $t = (n+1)\Delta t$. In the case of a negative answer, we modify the assumed contact point i_0 by passing to a neighboring point and repeat the process by imposing again the boundary conditions (9), (10) with the new value of i_0 and solving the resulting equations for the displacements on the boundary. This iterative process is continued until all the requirements as well as the boundary conditions are satisfied simultaneously yielding the correct contact region. The boundary conditions (3), (4) for a perfect adhesion are treated similarly.

APPLICATIONS

In the following we apply the proposed method of solution to the problem of indentation of a half-space by a wedge-shaped punch and parabolic punch. In some situations analytical solutions are known which can be employed in order to assess the accuracy and reliability of the numerical method. All the results presented in this paper were obtained with the spatial increments $\Delta x = \Delta y = d/50$, where d is a reference measure of length and the time increment $c_1\Delta t/d = 0.01$.

(1) *Smooth indentation by a wedge at a uniform speed*

Consider a rigid wedge-shaped die which indent the half-space at a given constant speed V. It is assumed that the indentation is smooth so that the boundary conditions are given by (2), (3) with (5) and $p(t) = Vt$.

By employing the self-similar method of solution, Robinson and Thompson[2], obtained an analytical solution for the normal stress σ_{yy} at the surface of the half-space for the different cases of subsonic, transonic and supersonic indentation assuming that the contact velocity $\alpha = \dot{X}(t)$ is constant.

In the subsonic case in which $\alpha < c_R$, with c_R being the speed of Rayleigh waves, the contact surface stress beneath the punch is given by

$$\sigma_{yy}(x,0,t) = -A \operatorname{arc\,cosh} \alpha t/|x| \quad |x| < \alpha t \tag{18}$$

with

$$A = 4\mu(1 - c_2^2/c_1^2)\tan\theta/\pi.$$

Furthermore, in (18), the contact velocity α is determined by the relation[11]

$$\int_0^\infty [1/c_1^2 + m^2]^{1/2}/[(1/\alpha^2 + m^2)^{1/2}R(-m^2)]\,dm = \pi c_2^2 V/[4(1 - c_2^2/c_1^2)\tan\theta] \tag{19}$$

with

$$R(-m^2) = (1/c_2^2 + 2m^2)^2 - 4m^2(1/c_1^2 + m^2)^{1/2}(1/c_2^2 + m^2)^{1/2}.$$

By a numerical integration of (19), the value of α for a given wedge angle θ and the elastic constants of the half-space can be determined.

In Fig. 2(a) the numerical and analytical solutions for $\sigma_{yy}(x,0,t)$ versus x/d are shown when $c_1t/d = 0.5$ and 1. The material is characterized by $(c_2/c_1)^2 = 0.3$ and the velocity of the indentation is chosen to be $V/c_2 = 0.05$ (i.e. $V/c_1 = 0.0274$) with $\tan\theta = 0.1$. The contact velocity is computed from (19) yielding $\alpha/c_1 = 0.207$ (whereas $\alpha_0 = \dot{X}_0(t) = V/\tan\theta = 0.274c_1$). It is well seen that a good correspondence between the two solutions exists.

In Fig. 2(b) we exhibit the numerical solution for the vertical displacement at the surface of the half-space $v(x,0,t)$ versus x/d at the same times. The contact region between the wedge and the half-space can be easily indentified in the figure from the plots of the stresses as well as the displacements. The contact velocity can be obtained from the plots of the numerical solution yielding the value $\alpha/c_1 = 0.19$ which is in a satisfactory agreement with the analytically predicted value mentioned before.

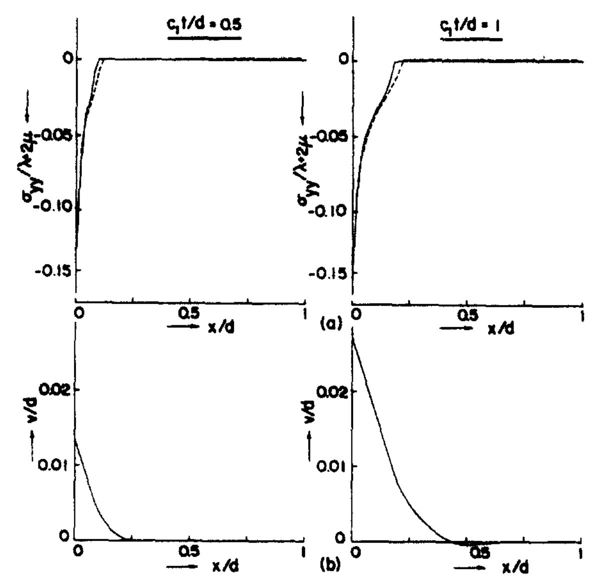

Fig. 2(a). Numerical solution (solid lines) and analytical solution (dashed lines) for the contact stresses induced by the uniform frictionless indentation by a wedge-shaped die. The boundary conditions are given by (2), (3) with (5) and $(c_2/c_1)^2 = 0.3$, $V/c_2 = 0.05$, $\tan\theta = 0.1$. (b) Numerical solution for the vertical displacements at the surface of the half-space.

The analytical solution (18) yields a singularity for σ_{yy} at $x = 0$ due to the existence of the corner of the wedge. The numerical solution, on the other hand, furnishes finite values for the stresses at $x = 0$ as can be expected. This is similar to the behavior obtained in crack propagation problems where the finite values of the computed stress can be shown to correspond to the asymptotic solution near the tip of the crack and therefore can be employed in order to extract the stress intensity factor of the crack, see [12].

It is obvious that the stresses and displacements can be produced easily for other points in the half-space.

When the indentation velocity of the punch V increases, the contact velocity α increases too. For $c_R < \alpha < c_1$ transonic indentation takes place for which the previous solution (18) is not valid. In Fig. 3 the numerical solution for the surface normal stress and vertical displacement is shown for $(c_2/c_1)^2 = 0.3$, $\tan\theta = 0.1$ and $V/c_2 = 0.13$ (i.e. $V/c_1 = 0.0712$). The analytical solution in this case is complicated and involves numerical integrations, but the velocity of contact is still given by (19). Equation (19) yields the value $\alpha/c_1 = 0.735$ whereas the numerical solution yields $\alpha/c_1 = 0.71$ indicating good agreement for this case too. It should be noted that $\alpha_0 = V/\tan\theta = 0.712c_1$ which is very close to the actual contact velocity.

When the indentation velocity V further increases such that the contact velocity α becomes greater than c_1, supersonic indentation occurs in which no disturbances can propagate more rapidly than the boundary of the region of contact. Hence, any point on the surface of the wedge would come into contact with the boundary of the half-space at the instant at which it crossed the original position on the surface $y = 0$. Accordingly, there will be no deformation of the surface at points beyond the region of contact so that $\alpha = \alpha_0$. Obviously, this advanced information facilitates significantly the analytical treatment.

In the numerical solution we need not utilize this advanced information but start the iterative process by assuming, as described in the previous section, that $\alpha = \alpha_0$ and then proceed with iterative procedure. The numerical results indicate that $\alpha = \alpha_0$ is in fact the actual contact velocity and no further iteration is needed. This remark is of importance since we need not distinguish in the numerical procedure between subsonic, transonic and supersonic indentations and the algorithm is unified and common for all cases. Indeed in cases when non-uniform indentation of a punch takes place, it might happen that $\alpha = \alpha(t)$ and some or all three possiblities occur at different stages of the penetration.

An analytical expression due to Robinson and Thompson [2], for the surface normal stress $\sigma_{yy}(x, 0, t)$ in the case of the supersonic indentation by a wedge exist. The explicit expression however, is lengthy and is not given here. It can be found in [11] where some obvious typographical errors should be corrected.

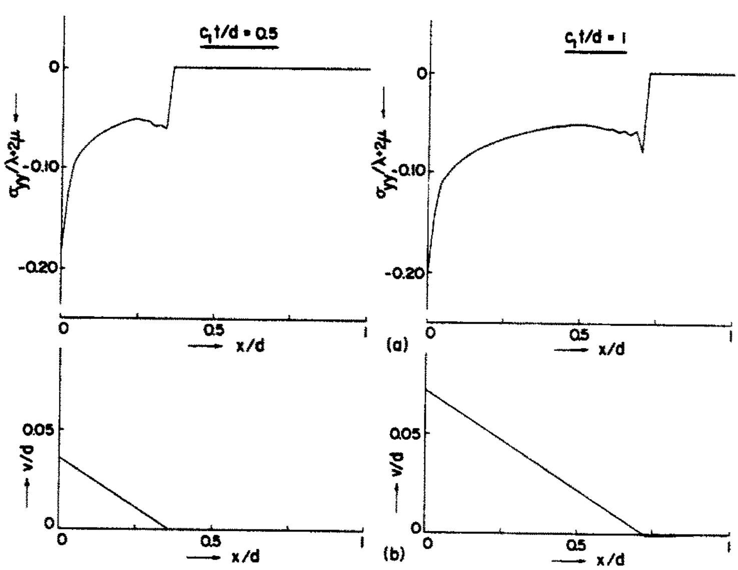

Fig. 3(a). Numerical solution for the contact stresses induced by the uniform frictionless indentation by a wedge-shaped die. The boundary conditions are given by (2), (3) with (5) $(c_2/c_1)^2 = 0.3$, $V/c_2 = 0.13$, $\tan\theta = 0.1$. (b) Numerical solution for the vertical displacements at the surface of the half-space.

In Fig. 4 the numerical and analytical solutions for $\sigma_{yy}(x, 0, t)$ are shown for $V/c_1 = 0.12$, $\tan\theta = 0.1$ and $(c_2/c_1)^2 = 0.3$ indicating satisfactory agreement. The value of the contact velocity as readily obtained from the numerical solution is $\alpha/c_1 = 1.2$. The numerical solution for the vertical displacement is also shown exhibiting the instantaneous penetration of the punch, as can be expected, and the displacement beyond the contact region vanishes (in contrast to the previous cases where non-zero displacements were obtained outside the contact region).

(2) *Frictional indentation by a wedge at a uniform speed*

The problem of a rigid wedge-shaped die which indent the half-space at a constant speed V assuming a perfect adhesion between the indentor and the half-space when they are in contact (i.e. a slip at the interface is completely prevented), was studied by Bedding and Willis in [4] and [5] for the subsonic and supersonic cases respectively. Each case requires a different analytical treatment. The relevent boundary conditions are given by (3)–(4) with (5) and $p(t) = Vt$.

In Fig. 5 we present the numerical solution for the surface normal stress $\sigma_{yy}(x, 0, t)$ and vertical displacement $v(x, 0, t)$ versus x/d for $(c_1/c_2)^2 = 3$, $\tan\theta = 0.1$ and $V/c_2 = 0.06$ (i.e. $V/c_1 = 0.0346$). This solution is also contrasted in the figure with the frictionless case. It is clear that both cases are very close so that the effect of the adhesion is minor. The contact velocities are subsonic and have the values $\alpha/c_1 = 0.278$ and 0.288 for the frictional and smooth indentors respectively. These values are taken from the plots presented in [4]. Our numerical solutions yield the corresponding values $\alpha/c_1 = 0.26$ and 0.27 which indicate again good agreement with the analytical derivations in [4]. Since the iterations involved in the numerical process start from the value $\alpha_0 = V/\tan\theta = 0.346c_1$, it turns out that several iterations are needed here in order to obtain the correct solution.

It should be noted that an analytical treatment for the indentation problem of a wedge in the transonic region under the assumption of a perfect adhesion between the indentor and the half-space is not yet known, see [5], whereas a numerical solution for this case can be easily obtained.

(3) *Frictionless indentations of a wedge at a time-dependent contact velocity*

We assume that the wedge indents the half-space with the time-dependent velocity $V(t)$. The boundary conditions are given by (2), (3) with (5) and $p(t) = 0.5at^2$, where a is a parameter, so that $V(t) = at$. In this case the contact velocity $\alpha = \dot{X}(t)$ will not be a constant as in the previous cases and an analytical treatment of this problem is not known.

In Fig. 6 the surface stresses and displacements are shown for $ad/c_1^2 = 0.08$, $\tan\theta = 0.1$ and $(c_2/c_1)^2 = 0.3$. The initial iteration for the contact velocity $\alpha_0 = \dot{X}_0(t)$ is given by $\alpha_0 = 0.5at/\tan\theta$, which is a linearly increasing function of time. This figure illustrates, therefore, the

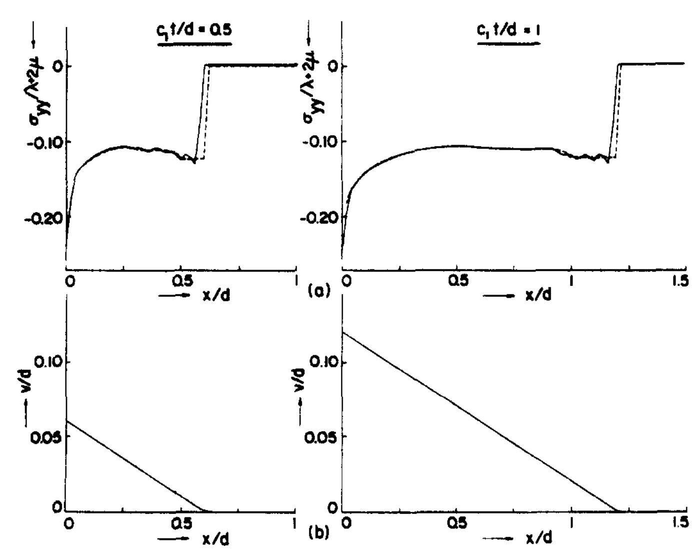

Fig. 4. Same as Fig. 2 but with $V/c_1 = 0.12$.

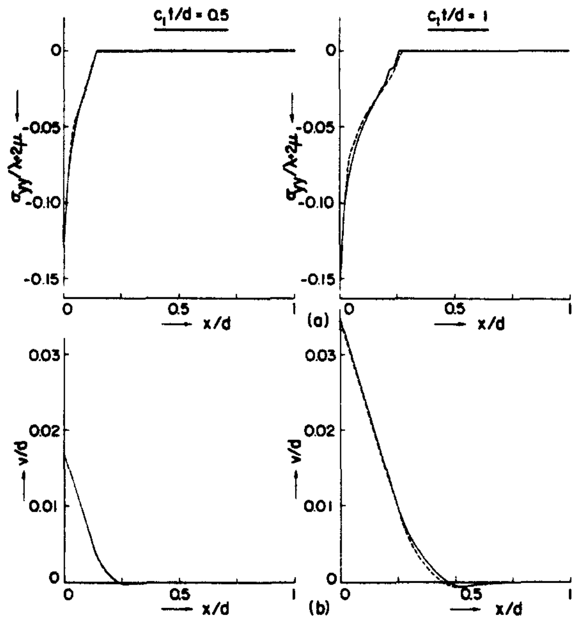

Fig. 5(a). Numerical solution for the contact stresses induced by the uniform frictional (solid lines) and smooth (dashed lines) indentation by a wedge-shaped die. The boundary conditions are given, respectively, by (3), (4) and (2), (3) with (5) and $(c_1/c_2)^2 = 3$, $V/c_2 = 0.06$, $\tan\theta = 0.1$. (b) The corresponding numerical solutions for the vertical surface displacements.

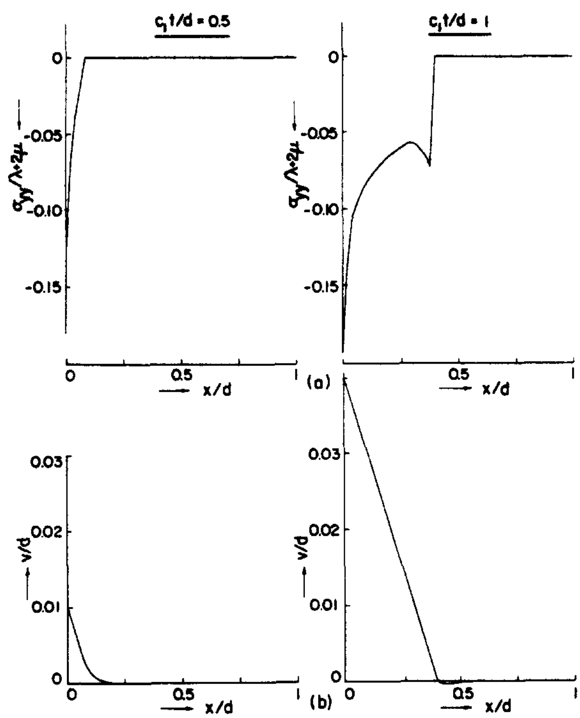

Fig. 6(a). Numerical solution for the contact stresses induced by the frictionless non-uniform indentation by a wedge-shaped die. The boundary conditions are given with (2), (3) with (5) and $p(t) = 0.5at^2$, $ad/c_1^2 = 0.8$, $\tan\theta = 0.1$, $(c_2/c_1)^2 = 0.3$. (b) Numerical solution for the vertical displacements at the surface of the half-space.

case of a varying contact velocity which has not been treated before. Due to the varying contact velocity, the character of the indentation will change in time from subseismic to superseismic.

(4) *Frictionless indentation by a parabolic punch*

For the dynamic frictionless contact problem of a parabolic punch with a half-space, the relevant boundary conditions are given by (2), (3) with (6). In the case of a uniformly accelerated punch for which $p(t) = 0.5at^2$, so that the indentation velocity $V(t) = at$ varies linearly with time, it turns out that the contact velocity $\alpha = \dot{X}(t)$ is constant and an analytical treatment is given by Cherepanov and Afanasev[6] for the subsonic case. This analytical solution will be employed as another test case to our numerical procedure.

According to [6], a closed form expression for σ_{yy} at the surface is given by

$$\sigma_{yy}(x,0,t) = -B[\alpha t)^2 - x^2]^{1/2}, \quad |x| < \alpha t \tag{20}$$

where B is a constant.

In Fig. 7 the numerical solution for the stress $\sigma_{yy}(x,0,t)$ and displacement $v(x,0,t)$ is shown versus x for $(c_2/c_1)^2 = 0.3$, $ad/c_1^2 = 0.05$ and $bd = 0.4$ when $c_1t/d = 0.5$ and 1. In the same figure the analytical solution (20) for σ_{yy} is also produced. The constant B in (20) was determined such that the value of σ_{yy} at $x = 0$ and $c_1t/d = 1$ as obtained by the numerical solution coincide with the analytical value at the same location and time. Its value is $Bd/\lambda + 2\mu = 0.18$. The constant contact velocity α was determined directly from the numerical solution and has the value $\alpha/c_1 = 0.22$ as against the higher value $\alpha_0 = [0.5a/b]^{1/2} = 0.25c_1$. Since B is a constant of the problem, once it has been determined, we can immediately utilize it in the solution (20) at any location beneath the punch and at any time.

It is clearly seen from the figure that good correspondence between the two solutions exist. The contact region can be clearly distinguished from the graphs of σ_{yy} and v. In the present case of a parabolic punch no singularity in σ_{yy} is obtained due to the absence of a corner. It is clear that solutions for contact velocities which are not in the subsonic domain as well as solutions for frictional parabolic punches can be obtained easily.

CONCLUSIONS

A numerical treatment to the problem of the two-dimensional dynamic indentation of a linearly isotropic elastic half-space by a rigid punch has been presented. The numerical procedure is based on an iterative process which is applied to the boundary conditions and is continued until the correct solution is obtained. The method is applicable to general shaped punch and time-dependent indentation velocities. Consequently, time-dependent contact velo-

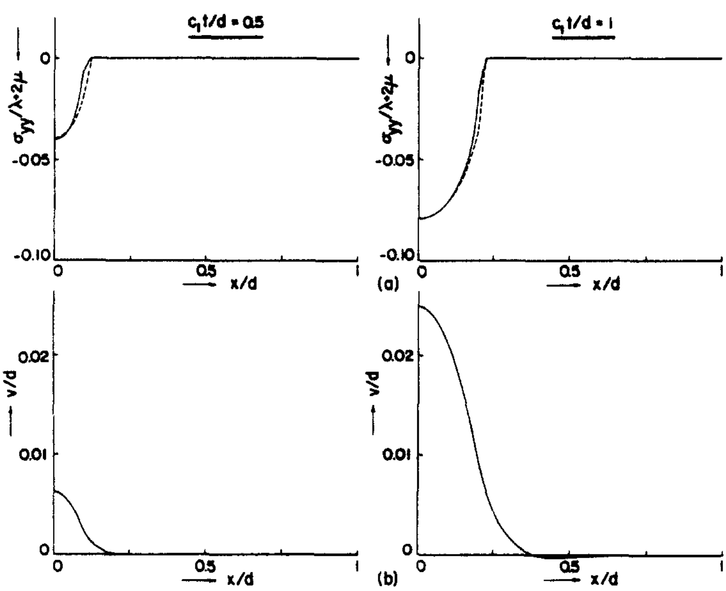

Fig. 7(a). Numerical solution (solid lines) and analytical solution (dashed lines) for the contact stresses induced by the frictionless indentation by a uniformly accelerated parabolic punch. The boundary conditions are given by (2), (3) with (6) and $p(t) = 0.5at^2$, $ad/c_1^2 = 0.05$, $bd = 0.4$, $(c_2/c_1)^2 = 0.3$. (b) Numerical solution for the vertical displacements at the surface of the half-space.

cities can be obtained. The numerical method of solution is examined in several situations where analytical results are known and good correspondence between the analytical and numerical results is obtained. Subsonic as well as transonic and supersonic indentation are treated in the same fashion. Solutions for perfectly frictional and smooth indentations can be obtained at any point within the half-space.

Possible generalizations of the method can be performed to: (1) The dynamic indentation of an elastic anisotropic half-space. (2) Combined normal and tangential loading of the half-space by the punch. (3) Indentation by axially symmetrical punches.

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REFERENCES

1. J. J. Kalker, Aspects of contact mechanics. In *The Mechanics of the Contact Between Deformable Bodies* (Edited by A. D. de Pater and J. J. Kalker), pp. 1-25. Delft University Press, Delft (1975).
2. A. R. Robinson and J. C. Thompson, Transient stresses in an elastic half-space resulting from the frictionless indentation of a rigid wedge-shaped die. *Z. angew. Math. Mech.* **54**, 139–144 (1974).
3. A. R. Robinson and J. C. Thompson, Transient stresses in an elastic half-space resulting from the frictionless indentation of a rigid symmetric conical die. *Proc. Camb. Phil. Soc.* **76**, 369–379 (1974).
4. R. J. Bedding and J. R. Willis, The dynamic indentation of an elastic half-space. *J. Elasticity* **3**, 289–309 (1973).
5. R. J. Bedding and J. R. Willis, High speed indentation of an elastic half-space by conical or wedge-shaped indentors. *J. Elasticity* **6**, 195–207 (1976).
6. G. P. Cherepanov and E. F. Afanasev, Some dynamic problems of the theory of elasticity—A review. *Int. J. Engng Sci.* **12**, 665–690 (1974).
7. A. R. Robinson and J. C. Thompson, Transient disturbances in a half-space during the first stage of frictionless indentation of a smooth rigid die of arbitrary shape. *Quart. Appl. Math.* **33**, 215–223 (1975).
8. L. M. Brock, Symmetrical frictionless indentation over a uniformly expanding contact region—1. Basic analysis. *Int. J. Engng Sci.* **14**, 191–199 (1976).
9. J. Aboudi and Y. Weitsman, Impact—deflection by oblique fibers in sparsely reinforced composites. *Z. angew. Math. Phys.* **23**, 828–844 (1972).
10. R. S. Varga, *Matrix Iterative Analysis*. Prentice-Hall, Englewood Cliffs, N.J. (1962).
11. A. C. Eringen and E. S. Suhubi, *Elastodynamics*. Vol. 2 (Linear Theory). Academic Press, New York (1975).
12. J. Aboudi, Numerical solution of dynamic stresses induced by moving cracks. *Comp. Meth. Appl. Mech. Eng.* **9**, 301–316 (1976).